ONE-DIMENSIONAL INVERSE THERMOELASTICITY PROBLEMS

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The unique solvability of two one-dimensional inverse problems on determining the coefficients of thermoelasticity equations is established; the solution algorithm for these problems is indicated and numerical calculations are given.

In the present paper we consider two one-dimensional inverse problems concerning the definition of the coefficients of the highest derivatives for thermoelasticity equations. The unknown elasticity and thermal conductivity coefficients are functions of one variable.

1. Let the thermoelasticity of a solid body be described by the system of equations

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = x^{-k}\frac{\partial}{\partial x}\left[x^k E(x)\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial x}(\beta T) + F_1(x,\ t), \tag{1}$$

$$c(x)\frac{\partial T}{\partial t} = x^{-k}\frac{\partial}{\partial x}\left[x^k \lambda(x)\frac{\partial T}{\partial x}\right] + F_2(x,\ t), \tag{2}$$

where $x \in (0, l)$, $t > 0$, $F_i(x, t)$, $i = 1, 2$, $\beta(x) > 0$, $\rho(x) > 0$, $c(x) > 0$ are prescribed continuous functions of their arguments; k, $l > 0$ are prescribed positive numbers; $E(x)$ and $\lambda(x)$ are unknown positive continuous functions; $u \equiv u(x, t)$, $T \equiv T(x, t)$ are solutions of Eqs. (1), (2). At $k=0$, $k=1$, $k=2$ system (1), (2) describes the thermoelasticity process in a segment, a cylinder, and a sphere, respectively.

For system (1), (2) we prescribe the initial and boundary conditions:

$$u|_{t=0} = \varphi_1(x),\ \left.\frac{\partial u}{\partial t}\right|_{t=0} = \varphi_2(x),\ u|_{x=0} = f_1(t),\ u|_{x=l} = f_2(t), \tag{3}$$

$$T|_{x=0} = T_1(t),\ T|_{x=l} = T_2(t),\ T|_{t=0} = T_0(x), \tag{4}$$

$$\left[x^k E(x)\frac{\partial u}{\partial x}\right]_{x=0} = g_1(t), \tag{5}$$

$$\left[x^k \lambda(x)\frac{\partial T}{\partial x}\right]_{x=0} = g_2(t), \tag{6}$$

where $\varphi_i(x)$, $f_i(t)$, $g_i(t)$, $T_i(t)$, $i = 1, 2$, $T_0(x)$ are prescribed continuous functions of their arguments.

The inverse problem on determining the coefficients for heat conduction equations and a wave equation separately in different formulations has been studied earlier (see [1-6] and others). In [7-9] special operating conditions are found when inverse thermoelasticity problems admit explicit solutions. For problem (1)-(6) these operating conditions are not satisfied; therefore, this calls for a separate consideration.

First we prove that the solution for the inverse problem (1)-(6) is uniquely defined. Actually, let us assume, on the contrary, that this problem has two solutions: the functions $u_i(x, t)$, $T_i(x, t)$, $E_i(x)$, $\lambda_i(x)$, $i = 1, 2$, are the solutions of problem (1)-(6). The discrepancy obtained below shows the validity of the statement about the unique solvalibity of problem (1)-(6). Now suppose $u = u_2 - u_1$, $T = T_2 - T_1$, $E = E_2 - E_1$, $\lambda = \lambda_2 - \lambda_1$. It is easy to verify that the functions u, T, λ, E satisfy the system conditions

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$$\rho(x)\frac{\partial^2 u}{\partial t^2} = x^{-k}\frac{\partial}{\partial x}\left[x^k E_1(x)\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial x}(\beta T) + x^{-k}\frac{\partial}{\partial x}\left[x^k E(x)\frac{\partial u_2}{\partial x}\right], \tag{7}$$

$$c(x)\frac{\partial T}{\partial t} = x^{-k}\frac{\partial}{\partial x}\left[x^k\lambda(x)\frac{\partial T}{\partial x}\right] + x^{-k}\frac{\partial}{\partial x}\left[x^k\lambda(x)\frac{\partial T_2}{\partial x}\right], \tag{8}$$

$$u|_{t=0} = \left.\frac{\partial u}{\partial t}\right|_{t=0} = u|_{x=0} = u|_{x=l} = 0, \tag{9}$$

$$\left[x^k E_1(x)\frac{\partial u}{\partial x}\right]_{x=0} = -\left[x^k E(x)\frac{\partial u_2}{\partial x}\right]_{x=0}, \tag{10}$$

$$T|_{t=0} = T|_{x=0} = T|_{x=l} = 0, \tag{11}$$

$$\left[x^k\lambda_1(x)\frac{\partial T}{\partial x}\right]_{x=0} = -\left[x^k\lambda(x)\frac{\partial T_2}{\partial x}\right]_{x=0}. \tag{12}$$

The inverse problem on determining the coefficient of the highest derivative $\lambda_1(x)$ in the heat conduction equation (2) has been investigated in [4, 5] and elsewhere. From the uniqueness theorem proved in these works it follows that in system (8), (11), (12) $T(x, t) \equiv 0$ and $\lambda(x) \equiv 0$. Then the second term in the right-hand side of Eq. (7) will be equal to zero. In this case the functions $u(x, t)$, $E(x)$ satisfy the equation

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = x^{-k}\frac{\partial}{\partial x}\left[x^k E_1(x)\frac{\partial u}{\partial x}\right] + x^{-k}\frac{\partial}{\partial x}\left[x^k E(x)\frac{\partial u_2}{\partial x}\right]$$

and conditions (9), (10). In this equation instead of t we take τ, multiply by $\tau/(2\sqrt{\pi t^3})\exp(-\tau^2/4t)$, and integrate the obtained equation with respect to τ in the domain $(0, \infty)$. Then the function

$$w(x, t) = \frac{1}{2\sqrt{\pi t^3}}\int_0^\infty \tau\exp\left(-\frac{\tau^2}{4t}\right)u(x, \tau)\,d\tau \tag{13}$$

satisfies the system conditions

$$\rho(x)\frac{\partial w}{\partial t} = x^{-k}\frac{\partial}{\partial x}\left[x^k E_1(x)\frac{\partial w}{\partial x}\right] + x^{-k}\frac{\partial}{\partial x}\left[x^k E(x)\frac{\partial w_2}{\partial x}\right], \tag{14}$$

$$w|_{t=0} = w|_{x=0} = w|_{x=l} = 0, \tag{15}$$

$$\left[x^k E_1(x)\frac{\partial w}{\partial x}\right]_{x=0} = -\left[x^k E(x)\frac{\partial w_2}{\partial x}\right]_{x=0}, \tag{16}$$

where $w_2 \equiv w_2(x, t)$ is the value of the integral in the right-hand side of (13) when we substitute $u_2(x, t)$ for $u(x, t)$. Consequently, the functions $E(x)$ and $w(x, t)$ obey the conditions of a system of the type (8), (11), (12), which was satisfied by the functions $T(x, t)$ and $\lambda(x)$. Because of this, from the results of [4, 5] it follows that $w(x, t) = E(x) = 0$. If in system (7), (9), (10) we assume that $E = T = 0$, then we obtain $u(x, t) \equiv 0$.

Thus, $u_1 \equiv u_2$, $T_1 \equiv T_2$, $E_1 \equiv E_2$, $\lambda_1 \equiv \lambda_2$, i.e., the solution of problem (1)-(6) is uniquely evaluated. We may easily check that problem (1)-(6) is, generally speaking, unstable. Examples of instability for the inverse heat conduction problem are presented in [1, 5, 6]. These examples remain valid also for problem (1)-(6), since here the function instability through Eqs. (1) exerts an effect on the definitions as well.

Remark. Under the conditions of problem (1)-(6) l may be ∞. Then, it is easy to see that the statement on the unique solvability of the problem remains valid, because in this case all the stages of its proof are retained.

2. In highly intensive processes of the thermal effect on materials the thermal characteristics depend substantially on the temperature distribution [3]. Let us now prescribe the equation

$$c(T)\frac{\partial T}{\partial t} = x^{-k}\frac{\partial}{\partial x}\left[x^k\lambda(T)\frac{\partial T}{\partial x}\right], \quad x>0,\ t>0, \tag{17}$$

instead of Eq. (2) in system (1)-(6), where $C(T) > 0$, $\lambda(T) > 0$ are continuous functions defined in $(-\infty, \infty)$. Furthermore, assume that instead of (4), (6) we prescribe the conditions

$$T|_{t=0} = 0, \; T|_{x=0} = T_0, \tag{18}$$

$$T|_{t=t_0} = \psi_0(x) \quad (\text{ or } \; T|_{x=x_0} = \psi_0(t)), \tag{19}$$

where $x_0 > 0$, $t_0 > 0$, $T_0 > 0$ are prescribed numbers; $\psi_0(x)$ (or $\psi_0(t)$) is prescribed function having the inverse $F(\psi_0)$.

Consider the inverse problem on determination of the functions $u(x, t)$, $T(x, t)$, $E(x) > 0$, $\lambda(T) > 0$ from conditions (1)-(3), (5), (17)-(19). Here the unknown coefficients are also sought in the class of positive continuous functions; $u(x, t)$, $T(x, t)$ are the classical solutions for the corresponding boundary-value problems. Equation (17) admits a self-similar solution [6] under conditions (18), (19). As was shown in [4], if the function $l_0(x)$ is prescribed, then we may define $\psi_0(t)$ by it and vice versa. Therefore, for the sake of definiteness suppose that the function $\psi_0(x)$ is prescribed.

In [4], with definite assumptions for the coefficient $\lambda(T)$ and the function $T(x, t)$ from the system of equations (17)-(19) the expressions

$$\lambda(T) = -F_T(T)\, F^{-k}(T) \int_0^T \frac{1}{2} c(s)\, F(s)\, F^k(s)\, ds, \tag{20}$$

$$T(x, t) = \psi_0\left(x \sqrt{\frac{t_0}{t}}\right). \tag{21}$$

are found. The right side of (20) is taken to be a positive continuous function.

If we substitute the function $T(x, t)$ from (21) into the right-hand side of (1), then we obtain the inverse problem (1), (3), (5) for defining the functions $E(x)$, $u(x, t)$. The unique solvability of this problem was discussed above in Sec. 1. Consequently, the solution of the inverse problem (1), (3), (5), (7), (19) is unique determined and the explicit conversion formulas (20), (21) are correct for $\lambda(T)$, $T(x, t)$.

3. We shall now consider an example of an inverse problem of the type (1)-(6), where the constant coefficients λ, E are sought:

$$\rho \frac{\partial^2 u}{\partial t^2} = E \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial T}{\partial x}, \tag{22}$$

$$\rho c \frac{\partial T}{\partial t} = \lambda \frac{\partial^2 T}{\partial x^2}, \; x \in R_1, \; t > 0, \tag{23}$$

$$u|_{t=0} = u_0, \; \left.\frac{\partial u}{\partial t}\right|_{t=0} = u_1, \tag{24}$$

$$T|_{t=0} = Q\delta(x), \tag{25}$$

$$E \left.\frac{\partial u}{\partial x}\right|_{\substack{x=b\\ t=0}} = -g_1, \tag{26}$$

$$\lambda \left.\frac{\partial T}{\partial x}\right|_{\substack{x=a\\ t=t_1}} = -g_2. \tag{27}$$

Equation (23) admits an explicit solution in the form of [10] under condition (25):

$$T(x, t) = \frac{1}{2} Q \sqrt{\frac{c\rho}{\pi\lambda t}} \exp\left(-\frac{\rho c x^2}{4\lambda t}\right). \tag{28}$$

Substituting (28) into (27), we obtain an algebraic equation for the definition of the thermal conductivity coefficient:

$$z e^{-z^2} = \varkappa, \tag{29}$$

here the following dimensionless quantities are introduced:

TABLE 1. Roots of Eq. (29) for Different Values of κ

κ	z_1	z_2	$\Delta z = z_2 - z_1$
0.250	0.26872	1.27705	1.00833
0.282	0.31055	1.12510	0.81455
0.400	0.52938	0.90135	0.37197
0.420	0.60741	0.81175	0.20434
0.425	0.64078	0.77558	0.13480
0.428	0.67527	0.73942	0.06415
0.42888	0.70711	0.70711	0

$$z = \frac{a}{2}\sqrt{\frac{\rho c}{\lambda t_1}}, \quad \varkappa = \frac{2g_2 t_1 \sqrt{\pi}}{Q\rho c}. \tag{30}$$

It is easy to verify that the maximum value of the function $f(z) = ze^{-z^2}$ is attained at $z = 1/\sqrt{2}$ and

$$\max f(z) = f\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2e}}.$$

This means that Eq. (29) has a solution only when $0 < \kappa \leq 1/\sqrt{2e}$. However, when $0 < \kappa < 1/\sqrt{2e}$ this equation has two solutions. This statement indicates the nonunique determination of the thermal conductivity coefficient in the case when the value of κ, calculated by formula (30), lies in the interval $(0, 1/\sqrt{2e})$.

Table 1 shows roots of Eq. (29) obtained for different values of κ. It can be seen from Table 1 that with increasing κ from zero to $1/\sqrt{2e}$, these roots approach each other and the difference between them approaches zero. When $\kappa = 1/\sqrt{2e} = 0.42888$ the roots coincide, i.e., when $\kappa = 1/\sqrt{2e}$ Eq. (29) has a unique solution, namely; $z_0 = 1/\sqrt{2} = 0.70711$.

Hence, the thermal conductivity coefficient λ is uniquely determined only in the case when

$$\frac{2g_2 t_1 \sqrt{\pi}}{Q\rho c} = \frac{1}{\sqrt{2e}}. \tag{30a}$$

Here, according to (30), we obtain

$$\lambda_0 = \frac{a^2 \rho c}{2t_1}. \tag{30b}$$

The elimination of $\rho c/t_1$ from expressions (30a), (30b) yields

$$\lambda_0 = \frac{g_2 a^2 \sqrt{2\pi e}}{Q}.$$

Substituting the value found for the thermal conductivity coefficient into (28), we determine the function $T(x, t)$ in terms of the initial data:

$$T(x, t) = \frac{Q}{a}\sqrt{\frac{t_1}{2\pi t}}\exp\left(-\frac{x^2 t_1}{2a^2 t}\right). \tag{31}$$

If we substitute (31) into (22), then we obtain the equation

$$\frac{1}{v_2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{Q\beta x}{Ea^3}\sqrt{\frac{t_1^3}{2\pi t^3}}\exp\left(-\frac{x^2 t_1}{2a^2 t}\right), \tag{32}$$

where $v = \sqrt{E/\rho}$ is the propagation velocity of expansion waves in an elastic medium.

TABLE 2. Roots of Eq. (37) for Different Values of q

q	0	1	2	5	10	10^2	10^3	10^4
ξ_0	0	1.07294	1.24940	1.76450	2.21328	3.93005	5.83674	7.84377

TABLE 3. Numerical Magnitudes of the Elasticity Modulus for Different Values of q

q	0	0.2	0.5	1	2	5	10	10^2
E_1	0	12.8	4.74	1.1512	1.56	3.1135	4.8986	15.445

By applying the Laplace transform to (32) with allowance for condition (24), we write

$$\frac{d^2\bar{u}}{dx^2}-\frac{p^2}{v^2}\bar{u}=-\frac{pu_0+u_1}{v^2}+\frac{\beta Qt_1\sqrt{t_1}}{Ea^2}\exp\left(-\frac{x}{a}\sqrt{2pt_1}\right). \tag{33}$$

The solution, bounded at infinity, for Eq. (33) will be

$$\bar{u}(x,\ p)=\frac{pu_0+u_1}{p^2}+\frac{\beta Qt_1v^2}{a^2Ep\left(\frac{2t_1v^2}{a^2}-p\right)}\exp\left(-\frac{x}{a}\sqrt{2pt_1}\right). \tag{34}$$

Passing to the original in (34), we obtain

$$\begin{aligned}u(x,\ t)=u_0+u_1t+\frac{\beta Q}{2E}\Bigg\{2\Phi^*\left(\frac{x}{a}\sqrt{\frac{t_1}{2t}}\right)-\exp\left(\frac{2v^2tt_1}{a^2}\right)\times\\ \times\left[\exp\left(\frac{2vxt_1}{a^2}\right)\Phi^*\left(\frac{x}{a}\sqrt{\frac{t_1}{2t}}+\frac{v}{a}\sqrt{2tt_1}\right)+\exp\left(-\frac{2vxt_1}{a^2}\right)\times\right.\\ \left.\times\Phi^*\left(\frac{x}{a}\sqrt{\frac{t_1}{2t}}-\frac{v}{a}\sqrt{2tt_1}\right)\right]\Bigg\}.\end{aligned} \tag{35}$$

Here

$$\Phi^*(\alpha)=1-\frac{2}{\sqrt{\pi}}\int_0^\alpha e^{-y^2}dy.$$

Hence we shall have

$$E\left.\frac{\partial u}{\partial x}\right|_{\substack{t=0\\ x=b}}=-\frac{Q\beta vt_1}{a^2}\operatorname{sh}\frac{2vbt_1}{a^2}. \tag{36}$$

The algebraic equation for determining v is obtained from (26) and (36):

$$\operatorname{sh}\xi=\frac{q}{\xi}. \tag{37}$$

In this case the following dimensionless quantities are introduced:

$$\xi=\frac{2vbt_1}{a^2},\quad q=\frac{2bg_1}{\beta Q}. \tag{38}$$

Equation (37) at any prescribed value of q has a unique solution. The corresponding solutions of Eq. (37) for various values of q are given in Table 2.

By the solution ξ_0 of Eq. (37) found, the propagation velocity of the elastic expansion wave v is determined from the formula

$$v = \frac{a^2\xi_0}{2bt_1}. \tag{39}$$

Table 3 presents the dependence of $E_1 = 4b^2t_1^2E/\rho a^4$ on q with account of Table 2. Hence we determine the elasticity modulus:

$$E = \frac{\rho a^4\xi_0^2}{4b^2t_1^2}. \tag{40}$$

Substituting (39) and (40) into (35), we define the bias field via the initial data

$$\begin{aligned} u(x,\ t) = u_0 + u_1 t + \frac{\beta Q b^2 t_1^2}{a^4 \rho \xi_0^2}\Bigg\{2\Phi^*\left(\frac{x}{a}\sqrt{\frac{t_1}{2t}}\right) - \exp\left(\frac{a^2\xi_0^2 t}{2b^2 t_1}\right) \times \\ \times \left[\exp\left(-\frac{x}{b}\xi_0\right)\Phi^*\left(\frac{x}{a}\sqrt{\frac{t_1}{2t}} - \frac{a\xi_0}{b}\sqrt{\frac{t}{2t_1}}\right) + \exp\left(\frac{x}{b}\xi_0\right) \times \right. \\ \left. \times \Phi^*\left(\frac{x}{a}\sqrt{\frac{t_1}{2t}} + \frac{a\xi_0}{b}\sqrt{\frac{t}{2t_1}}\right)\right]\Bigg\}. \end{aligned} \tag{41}$$

Thus, we have found explicit expressions for the unknown coefficients λ and E. By these quantities the expressions for the functions T(x, t) and u(x, t) are determined. If instead of the additional conditions (25), (27) we take the conditions

$$u\Big|_{\substack{x=b\\t=0}} = f_1,\quad T\Big|_{\substack{x=a\\t=t_1}} = f_2, \tag{42}$$

where f_1 and f_2 are prescribed numbers, then, using them similarly to the above, we can solve the inverse problem on the definition of λ, E, u(x, t), T(x, t). Furthermore, if the coefficients Q, β, c, ρ are unknown, then, taking additionally conditions of the type (25), (27), (42), it is also possible to determine them. In the special case when λ and E are prescribed, some of the unknown coefficients may be defined from the expressions found above for and E.

REFERENCES

1. M. M. Lavrent'yev, V. G. Romanov, and S. P. Shishatskii, Incorrect Problems of Mathematical Physics and Analysis [in Russian], Moscow (1980).
2. A. N. Tikhonov and A. A. Samarskii, Equations of Mathematical Physics [in Russian], Moscow (1972).
3. A. D. Kovalenko, Fundamentals of Thermoelasticity [in Russian], Kiev (1970).
4. I. B. Basovich, Prikl. Mat. Mekh., **38**, No. 3, 376-383 (1974).
5. A. D. Iskenderov, Izv. Akad. Nauk AzSSR, No. 2, 14-21 (1971).
6. A. D. Iskenderov, "Inverse Problems for Parabolic and Elliptic Equations," Author's Abstract of Doctoral Dissertation, Moscow (1978).
7. A. D. Iskenderov, T. B. Gardashov, and Sh. M. Bakhyshev, Inzh.-Fiz. Zh., **58**, No. 2, 326-327 (1990).
8. Sh. M. Bakhyshev, Dokl. Akad. Nauk AzSSR, **25**, No. 2, 3-6 (1989).
9. V. A. Ditkin and A. P. Prudnikov, Operational Calculus [in Russian], Moscow (1975).
10. A. V. Luikov, Heat Conduction of Solid Bodies [in Russian], Moscow (1967).